Introduction to Coleman Integration

Notation:
- $C$ hyperelliptic curve over an unramified extension $k$ of $\mathbb{Q}_p$ with $p$ a prime of good ordinary reduction
- Points $P, Q, R$ on $C$
- Differential forms $\omega, \omega'$ of the second kind on $C$
- Differential forms $\omega_1, \ldots, \omega_{2g-1}$, a basis for $H^0_{\text{dR}}(C)$, where $\omega_i = \frac{\alpha_i}{T}$

Coleman constructed a definite integral with the following properties:
1. Linearity: $\int_{P}^{Q}(\alpha \omega + \beta \omega') = \alpha \int_{P}^{Q} \omega + \beta \int_{P}^{Q} \omega'$
2. Additivity: $\int_{P}^{Q} \omega = \int_{P}^{S} \omega + \int_{S}^{Q} \omega$
3. Change of variables: If $C'$ is another curve and $\phi: C \to C'$ a rigid analytic map between wide opens then $\int_{P}^{Q} \phi^* \omega = f(\phi) \int_{P}^{Q} \omega$
4. Fundamental theorem of calculus: $\int_{S}^{P} df = f(Q) - f(P)$

"Tiny" Integrals

Suppose $P, Q \in C(\mathbb{C}_p)$ are in the same residue disc. We compute $\int_{P}^{Q} \omega$ locally:
1. Construct an interpolation $x(t), y(t)$ from $P$ to $Q$.
2. Formally integrate the power series in $t$: $\int_{P}^{Q} \omega = \int_{0}^{1} x(t) \omega(x) + \int_{0}^{1} y(t) \omega(y) dt$.

Integrals via Kedlaya’s algorithm

If $P, Q$ are in different residue discs, we use Frobenius $\phi$ to construct $\int_{P}^{Q} \omega$:
1. Find Teichmüller points $P', Q'$ in the discs of $P, Q$.
2. Compute the tiny integrals $\int_{P'}^{Q'} \omega$, $\int_{P'}^{R'} \omega$.
3. Calculate the action of Frobenius on each basis element $\phi^* \omega = dt + \sum \pm \omega_j M_j \omega_j$.
4. Change of variables gives $\sum \pm (M - 1) \int_{P}^{Q} \omega = f(P) - f(Q')$, and solving the linear system gives the integrals $\int_{P}^{Q} \omega$.
5. Correct endpoints to recover $\int_{P}^{Q} \omega = \int_{P'}^{Q'} \omega + \int_{P'}^{R'} \omega + \int_{R'}^{Q'} \omega$.

Application: Coleman-Gross height pairing

The Coleman-Gross height pairing is a symmetric bilinear pairing $h: \text{Div}^0(C) \times \text{Div}^0(C) \to \mathbb{Q}_p$, which can be written as a sum of local height pairings $h = \sum_v h_v$ over all finite places $v$ of the number field $K$.

Local height above $P$

Let $D_1, D_2 \in \text{Div}^0(C)$ have disjoint support and $\omega_v$ be a normalized differential associated to $D_v$. The local height pairing at $v$ above $P$ is given by

$h_v(D_1, D_2) = \text{tr}_{K_v} \left( \int_{D_1} \omega_v \right)$.

To construct $h_v$:
1. Choose a differential $\omega$ with $\text{Res}(\omega) = D_v$.
2. Fix a splitting $\hat{H}^0_{\text{dR}}(C/k) = \hat{H}^0_{\text{dR}}(C/k) \oplus W$, where $W$ is the unit root subspace for the action of Frobenius.
3. Via the canonical homomorphism $\psi: T(k) \to T(k)$, compute $\hat{H}^0_{\text{dR}}(C/k)$, compute $\hat{W}(\omega) = \eta + \psi(D_v)$, for $\eta$ holomorphic. Then $\omega_v := \omega - \eta$.

Coleman integration: meromorphic differential

Let $\phi$ be a $p$-power lift of Frobenius and set $\alpha := \phi^* \omega - \omega$. Then for a differential with residue divisor $D = (R) - (S)$, we compute

$\int_{D} \omega = \int_{S}^{R} \omega = \frac{1}{1 - p} \left( \int_{S}^{R} \alpha \omega + \sum \text{Res} \left( \alpha \int_{R}^{S} \beta \right) \right) - \frac{1}{1 - p} \int_{S}^{R} \omega + \int_{R}^{S} \omega$.

Example: global $p$-adic heights for genus 1

Example: Let $C$ be the elliptic curve $y^2 = x^3 - 5x$, with $Q = (-1, -2)$, $Q' = (1, -1)$. Then for $Q = (1, 1)$, $R = (5, 10)$, $R' = (5, -10)$, so that $(Q) - (Q') = (R) - (R') = (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}})$. We verify the 13-adic height of $P$.

Above 13, the local height $h_{13}(Q) - (Q')$, $(R) - (R')$ is $2 \cdot 13 + 6 \cdot 13^3 + 5 \cdot 13^4 + O(13^5)$.

Away from 13, the only nontrivial contribution is $2 \log 3$ (by work of Müller).

So the global 13-adic height is $12 \cdot 13 + 4 \cdot 13^2 + 10 \cdot 13^3 + 9 \cdot 13^4 + O(13^5)$.

This example generalizes Mazur’s Tate-Teichmuller height in Sage:

sage: C = EllipticCurve([-5, 0])
sage: f = C.padic_height(13)
sage: f(C(9/4, -3/8)) + O(13^5)
12 + 4*13^2 + 10*13^3 + 9*13^4 + O(13^5)

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