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The Hodge and Tate conjectures: some numerical experiments.

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(joint with
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July 23, 2010
Algebraic cycles and the cycle class map

\[ V = \text{smooth, projective variety over } \mathbb{C} \text{ of dimension } d. \]

\[ \text{CH}^j(V) = \left\{ \begin{array}{c}
\text{Codimension } j \\
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\end{array} \right\} \otimes \mathbb{Q} / \sim, \]

where \( \sim \) denotes rational equivalence.

The cycle class map:

\[ \text{cl} : \text{CH}^j(V) \longrightarrow H^{j,j}_{dR}(V(\mathbb{C})) \cap H^{2j}_{B}(V(\mathbb{C}), \mathbb{Q}). \]

\[ \langle \text{cl}(\Delta), \alpha \rangle = \int_{\Delta(\mathbb{C})} \alpha, \quad \forall \alpha \in H^{2d-2j}_{dR}(V(\mathbb{C})). \]
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Conjecture (Hodge Conjecture)

*The cycle class map $\text{cl}$ is surjective.*

A cohomology class in $H^j_{dR}(V(\mathbb{C})) \cap H^j_B(V(\mathbb{C}), \mathbb{Q})$

—i.e, a class of type $(j,j)$ with rational periods—

is called a *Hodge cycle*.

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*Every Hodge cycle is the class of an algebraic cycle.*

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The Tate conjecture

The cycle class map has an analogue in \( \ell \)-adic étale cohomology:

\[
cl_\ell : \text{CH}^i(V/F) \otimes \mathbb{Q}_\ell \longrightarrow H^{2j}_{\text{et}}(V_{\overline{F}}, \mathbb{Q}_\ell(j))^{G_F}.
\]

Conjecture (Tate)

*The \( \ell \)-adic cycle class map is surjective.*

The Hodge conjecture is known for surfaces, and for codimension one cycles, but there seems to be very little evidence for cycles of higher codimension.
The challenge

André Weil (Collected works, 1979).

La question que pose la “conjecture de Hodge” est bien naturelle... Par malheur, en dépit du mot de “conjecture”, il n’y a, que je sache, pas l’ombre d’une raison d’y croire; on rendrait service aux géomètres si l’on pouvait trancher la question au moyen d’un contre-exemple.

A challenge for the experimental mathematician:

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CM elliptic curves

\[ D \in \{7, 11, 19, 43, 67, 163\}. \]

\[ K = \text{quadratic imaginary field of discriminant } -D. \]

\[ (K \text{ has class number one and } \mathcal{O}_K^\times = \pm 1.) \]

\[ A = \text{elliptic curve over } \mathbb{Q} \text{ of conductor } D^2 \text{ with } \text{End}_K(A) = \mathcal{O}_K. \]

\[ \omega_A \in \Omega^1(A/\mathbb{Q}): \text{the Néron differential of } A. \]

\[ \Lambda_A := \{ \int_\gamma \omega_A \} = \mathcal{O}_K \cdot \Omega_A. \]

\[ \Omega_A \in \mathbb{C} \text{ is called the } \text{Chowla-Selberg period} \text{ attached to } A. \]
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The de Rham cohomology of $A$

Because $A$ has CM, we can write

$$H^1_{\text{dR}}(A/K) = K \cdot \omega_A \oplus K \cdot \eta_A,$$

where

1. $\omega_A$ is the Néron differential, viewed as an element of $\Omega^1(A/K)$;

2. $\eta_A$ is the generator of $H^0_{\text{dR}}(A/\mathbb{C})$ normalised so that

$$\langle \omega_A, \eta_A \rangle = \frac{1}{2\pi i} \int_{A(\mathbb{C})} \omega_A \wedge \eta_A = 1.$$

The fact that $\eta_A$ is defined over $K$ is specific to the CM setting.
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The Hecke character attached to $A$

Theorem (Deuring)

There is a Hecke character $\psi_A$ of $K$ of infinity type $(1, 0)$ and conductor $\sqrt{-D}$ satisfying $L(A, s) = L(\psi_A, s)$.

More precisely, for all $a \in \mathcal{O}_K$,

$$\psi_A((a)) = \left( \frac{a}{\sqrt{-D}} \right) a.$$
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Fix an integer $r \geq 0$, and let $\psi = \psi_A^{r+1}$.

The character $\psi$ is of infinity type $(r + 1, 0)$.

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\text{conductor}(\psi) = \begin{cases} 
(\sqrt{-D}) & \text{if } r \text{ is even;} \\
1 & \text{if } r \text{ is odd.}
\end{cases}
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\theta_\psi = \frac{1}{2} \sum_{a \in \mathcal{O}_K} \psi(a) q^{a \bar{a}} \in \begin{cases} 
S_{r+2}(\Gamma_0(D^2)) & \text{if } r \text{ is even;} \\
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Kuga-Sato varieties

Modular forms in $S_{r+2}(\Gamma)$ give rise to cohomology classes on certain Kuga-Sato varieties.

Modular curve:

$$C = \begin{cases} 
Y_0(D^2) & \text{if } r \text{ is even;} \\
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$W'_r = r$-fold fiber product of the “universal” elliptic curve over $C$.

The open variety $W'_r$ admits a smooth compactification, called the Kuga-Sato variety attached to $(C, r)$, and denoted $W_r$. 
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so that \( C(\mathbb{C}) = \mathcal{H}/\Gamma \). Then

\[ W_r'(\mathbb{C}) = (\mathbb{Z}^{2r} \rtimes \Gamma) \backslash (\mathbb{C}^r \times \mathcal{H}), \]

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\[ (m_1, n_1, \ldots, m_r, n_r)(w_1, \ldots, w_r, \tau) = (w_1+m_1+n_1\tau, \ldots, w_r+m_r+n_r\tau, \tau), \]

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The holomorphic \((r + 1)\)-form on \(\mathbb{C}^r \times \mathcal{H}\)

\[ \omega_{\theta, \psi} = (2\pi i)^{r+1} \theta_{\psi}(\tau) dw_1 \cdots dw_r d\tau \]

is invariant under \(\mathbb{Z}^{2r} \rtimes \Gamma\), and hence corresponds to a regular \((r + 1)\)-form in \(\Omega^{r+1}(W'_r/\mathbb{C})\).

It extends to \(W_r\) (cuspidality) and is defined over \(\mathbb{Q}\) (\(q\)-expansion principle.)

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A family of Hodge cycles

The ambient variety: \( V_r := W_r \times A^{r+1} \), (of dimension \( 2r + 2 \)).

**Proposition**

The class \( \Phi_{\text{Hodge}} := \omega_{\theta, \psi} \wedge \eta_A^{r+1} \) is a Hodge cycle in \( H^{2r+2}_{dR}(V_r/\mathbb{C}) \).

This result is “consistent” with the \( \ell \)-adic picture:

- The \( \ell \)-adic representation \( V_{\theta, \psi} \) of \( G_{\mathbb{Q}} \) attached to the modular form \( \theta, \psi \) is a constituent of \( H^{r+1}_{et}(W_r/\overline{\mathbb{Q}}, \mathbb{Q}_\ell) \).
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A special case of the Hodge conjecture

Question

Given $D$ and $r \geq 0$, produce an algebraic cycle

\[ \Phi \subset V_r = W_r \times A^{r+1} \]  

such that \( \text{cl}(\Phi) = \Phi_{\text{Hodge}} \)

...or show that no such cycle exists!

The Tate conjecture suggests that the cycle class $\Phi$ can be defined over $K$. (Or even, with some care, over $\mathbb{Q}$...)
Some examples

- $r = 0$. Then, $V_r = C \times A$. The cycle $\Phi$ is the graph of a modular parametrisation $X_0(D^2) \to A$.

- $r = 1, D = 7$. The Kuga-Sato variety $W_1$ is an elliptic K3-surface with maximal Picard rank 20.

  **Shioda-Inose**: there exists an involution $\iota$ on $W_1$, such that

  $$W_1/\iota = \text{Kummer}(B) = B/\pm 1, \text{ with } B \sim A \times A.$$  

  The desired cycle can be built from the image of

  $$W_1 \times_{W_1/\iota} B \quad \text{in} \quad W_1 \times A^2.$$  

  An explicit equation (over $\mathbb{Q}$) for the Shioda-Inose structure in this case has been computed by Elkies...

- These are the only cases for which the algebraic cycle has been produced! $D = 163, r = 15$ ???
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- These are the only cases for which the algebraic cycle has been produced! $D = 163$, $r = 15$???
Let $X_r = W_r \times A^r$, and let

$$\text{CH}^{r+1}(X_r)_0 \subset \text{CH}^{r+1}(X_r)$$

denote the subgroup of classes of null-homologous cycles.

**Key Remark:** The conjectural algebraic cycle

$$\Phi \subset V_r = W_r \times A^{r+1} = X_r \times A$$

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$$\Phi : \text{CH}^{r+1}(X_r)_0 \longrightarrow \text{CH}^1(A)_0 = A,$$

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Definition of $\Phi$

1. Let $\pi_A : X_r \times A \to A$, $\pi_X : X_r \times A \to X_r$ be the natural projections to each factor.

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$$\dim(\pi_X^{-1}(\Delta)) = \dim(\Phi) = r + 1 = \frac{1}{2} \dim V_r;$$

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is well-defined.
The group $\text{CH}^{r+1}(X_r)_0(\bar{\mathbb{Q}})$ is infinitely generated.

Proof.

$\text{CH}^{r+1}(X_r)_0$ contains an infinite collection of explicit null-homologous cycles: the generalised Heegner cycles. C. Schoen: they generate a subgroup of infinite rank.

Generalised Heeger cycles: Indexed by $\varphi : A \rightarrow A'$.

$\Delta'_{\varphi,r} := \text{graph}(\varphi)^r \subset (A \times A')^r = (A')^r \times A^r \subset W_r \times A^r = X_r.$

$\Delta_{\varphi,r} := \varepsilon \Delta'_{\varphi,r}$,

where $\varepsilon$ is a simple projector that makes $\Delta_{\varphi,r}$ null-homologous.
Test cycles

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The conjecture on generalised Heegner cycles

The cycle $\Delta_{\varphi,r}$ is defined over a ring class field $H_{\varphi}$ attached to $\varphi$. 

**Proposition**

*If the cycle $\Phi$ exists, then* $\{\Phi(\Delta_{\varphi,r})\}_{\varphi}$ *generates an infinite rank subgroup of* $A(K^{ab})$.  
*In particular,* $\Phi(\Delta_{1,r})$ *belongs to* $A(K)$. 

The points $\Phi(\Delta_{\varphi,r})$ are called *Chow-Heegner points*. 

**Some problems:**

1. Calculate Chow Heegner points numerically *without* calculating the algebraic cycle $\Phi$ beforehand.  
2. Describe the exact position of $\Phi(\Delta_{1,r})$ in $A(K)$ as $r \geq 0$ varies.
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Recall the classical Abel-Jacobi map:

\[ \text{AJ}_A : \text{CH}^1(A(\mathbb{C}))_0 = A(\mathbb{C}) \longrightarrow \frac{\Omega^1(A/\mathbb{C})^\vee}{H_1(A(\mathbb{C}), \mathbb{Z})} = \mathbb{C}/\Lambda_A, \]

given by

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(If \( \Delta = P - Q \), then \( \partial^{-1}(\Delta) \) is any path from \( Q \) to \( P \).)

Higher-dimensional analogue (Griffiths-Weil):

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**Theorem**

If $\Phi$ exists, then

$$AJ_A(\Phi(\Delta_\varphi,r))(\omega_A) = AJ_{X_r}(\Delta_\varphi,r)(\omega_{\theta_\psi} \wedge \eta^r_A).$$

**Proof.**

Functoriality of Abel-Jacobi maps under correspondences:

$$AJ_A(\Phi(\Delta_\varphi,r))(\omega_A) = AJ_{X_r}(\Delta_\varphi,r)(\Phi^* \omega_A),$$

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Let $\Delta_{\varphi,r}$ be the generalised Heegner cycle on $X_r$ corresponding to the isogeny $\varphi : A \to A'$, and let $\tau \in \mathcal{H}$ satisfy $A'(\mathbb{C}) = \mathbb{C}/\langle 1, \tau \rangle$. Then

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Since $\theta_{\psi}$ is a modular form of weight $r + 2$, the expression on the right is an “incomplete Eichler integral”.

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The position of $\Phi(\Delta_{\varphi}, r)$ in $A(K)$

Conjecture (Bertolini-Prasanna-D)

If $D = 7$, then $\Phi(\Delta_1, r) = 0$. For all $D \in \{11, 19, 43, 67, 163\}$ and all odd $r \geq 1$, the Chow-Heegner point $\Phi(\Delta_1, r)$ belongs to $A(K) \otimes \mathbb{Q}$ and is given by the formula

$$\Phi(\Delta_1, r) = \sqrt{-D} \cdot m_r \cdot P_A,$$

where $m_r \in \mathbb{Z}$ satisfies the formula

$$m_r^2 = \frac{2r!(2\pi \sqrt{D})^r}{\Omega_{A}^{2r+1}} L(\psi_A^{2r+1}, r + 1),$$

and $P_A$ is the generator of $A(\mathbb{Q}) \otimes \mathbb{Q}$ given in the next slide.
The Mordell-Weil generators $P_A$

<table>
<thead>
<tr>
<th>$D$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_6$</th>
<th>$P_A$</th>
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<td>−107</td>
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<td>−</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>−1</td>
<td>1</td>
<td>−7</td>
<td>10</td>
<td>(4, 5)</td>
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<tr>
<td>19</td>
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<td>0</td>
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<td>−38</td>
<td>90</td>
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<td>0</td>
<td>1</td>
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<td>67</td>
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<td>0</td>
<td>1</td>
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<td>243528</td>
<td>$(\frac{201}{4}, \frac{-71}{8})$</td>
</tr>
<tr>
<td>163</td>
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<td>0</td>
<td>1</td>
<td>−2174420</td>
<td>1234136692</td>
<td>(850, −69)</td>
</tr>
</tbody>
</table>
Where does this conjecture come from?

\textbf{p-adic methods}

- The complex Abel-Jacobi map admits a \( p \)-adic analogue:
  \[ \text{AJ}_{X_r}^{(p)} : \text{CH}^{r+1}(X_r)_0(\mathbb{C}_p) \longrightarrow \text{Fil}^{r+1} H^{2r+1}_{dR}(X_r/\mathbb{C}_p)^\vee. \]

- Bertolini, Prasanna, D: A \( p \)-adic Gross-Zagier formula relating \( \text{AJ}_{X_r}^{(p)}(\Delta_{\varphi,r}) \) to \( p \)-adic \( L \)-functions.

- In the case at hand, this formula gives (for all odd \( r \geq 1 \))
  \[ \frac{\text{AJ}_{X_r}^{(p)}(\Delta_{1,r})(\omega_{\theta_{\psi}} \wedge \eta_A^r)}{\text{AJ}_{X_1}^{(p)}(\Delta_{1,r})(\omega_{\theta_{\psi}} \wedge \eta_A^r)} = \frac{m_r}{m_1}, \]

where
  \[ m_r^2 = \frac{2r!(2\pi \sqrt{D})^r}{\Omega_{\psi_A}^{2r+1}} L(\psi_A^{2r+1}, r + 1). \]
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The outcome of the experiment

Conjecture BDP was tested numerically to 200 digits of decimal accuracy, for all $D$ and all odd $1 \leq r \leq 15$.

Let $\tilde{P}_A \in \mathbb{C}$ be a lift of $P_A \in A(\mathbb{C}) = \mathbb{C}/\Lambda_A$.

The table in the next slide reproduces an integer $m_r$ (of relatively small height) satisfying

$$AJ_{X_r}(\Delta_{1,r}) = \sqrt{-D} \cdot m_r \cdot \tilde{P}_A \pmod{\Lambda_A},$$

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The numbers are in perfect agreement with a table of “square roots” of $L(\psi_A^{2r+1}, r + 1)$ produced by Villegas...
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<tr>
<td>1</td>
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Conclusion

All the numerical experiments are consistent with the existence of an algebraic cycle $\Phi \in \text{CH}^{r+1}(V_r)$ attached to $\Phi_{\text{Hodge}}$.

This gives some indirect evidence for the Hodge conjecture for some varieties of large dimension (up to 32, in the computed ranges.)

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Thank you for your attention!